Strong breakdown of equipartition in uniform granular mixtures

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Abstract. A binary mixture made of heavy and light inelastic hard spheres in the free cooling state is considered. First, the regions in the parameter space where the partial granular temperature of the heavy species is larger (or smaller) than that of the light species are analyzed. Next, the asymptotic behavior of the mean square velocity ratio in the disparate-mass limit is investigated, assuming different scaling laws for the parameters of the mixture. It is seen that two general classes of states are possible: a "normal" state and an "ordered" state, the latter representing a strong breakdown of energy equipartition.

Keywords: Granular fluid mixtures; Boltzmann equation; equipartition of energy; critical exponents
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INTRODUCTION

As is well known, granular fluids are intrinsically out of equilibrium. This clearly manifests itself in the case of granular mixtures, where the equipartition of energy is broken down (even in homogeneous and isotropic states), as shown by theory [1, 2], simulations [3], and experiments [4]. In this work we consider a binary mixture made of heavy (h) and light (ℓ) inelastic hard spheres in the free cooling state. By using kinetic theory tools, one can derive a tenth-degree algebraic equation [1] whose solution gives the mean square velocity ratio $\phi=\langle v_h^2\rangle/\langle v_\ell^2\rangle=T_h m_\ell/T_\ell m_h$ (which plays the role of an "order" parameter) as a function of the control parameters of the problem (the densities, the three coefficients of normal restitution, and the mass and size ratios). We first investigate the regions in the parameter space where the partial granular temperature of the heavy species is larger ($T_h>T_\ell$) or smaller ($T_h<T_\ell$) than that of the light species. We next analyze the asymptotic behavior of ϕ in the disparate-mass limit ($m_\ell/m_h\to 0$), assuming different scaling laws for the coefficients of restitution, size ratio, and concentrations. We observe that it is possible to distinguish two general classes of states: a *normal* state (where $\phi\to 0$) and an *ordered* state (where $\phi\neq 0$), the latter representing a strong breakdown of energy equipartition in which the temperature ratio T_h/T_ℓ diverges. These two classes can be further resolved. Thus, the normal state can be *partitioned* ($T_h/T_\ell=$ finite), *mono-energetic h* ($T_h/T_\ell\to\infty$), or *mono-energetic* ℓ ($T_h/T_\ell\to 0$). Moreover, if the concentration of heavy particles vanishes, the ordered state can become *super-ordered* ($\phi\to\infty$). The phase diagrams corresponding to this rich phenomenology are presented. In particular, in the limit of vanishing concentration of heavy particles, the partitioned phase is separated from the ordered and from the super-ordered phases by two critical lines. This extends a previous analysis made in the case of an impurity particle immersed in an inelastic gas [5].

BASIC EQUATIONS AND CONDITIONS FOR $T_h>T_\ell$

Let us consider the homogeneous cooling state of a binary mixture of two granular species (h and ℓ) characterized by mole fractions $x_h=n_h/n$ and $x_\ell=1-x_h$, masses m_h and $m_\ell<m_h$, diameters σ_h, σ_ℓ, and $\sigma_{h\ell}=(\sigma_h+\sigma_\ell)/2$, coefficients of normal restitution α_{hh}, $\alpha_{\ell\ell}$, and $\alpha_{h\ell}$, and pair correlation contact values χ_{hh}, $\chi_{\ell\ell}$, and $\chi_{h\ell}$. This gives 10 independent parameters. However, in appropriate reduced units the number of control parameters reduces considerably. The set of two Boltzmann equations are

$$\partial_t f_h(\mathbf{v})=J_{hh}[\mathbf{v}|f_h,f_h]+J_{h\ell}[\mathbf{v}|f_h,f_\ell],\quad \partial_t f_\ell(\mathbf{v})=J_{\ell h}[\mathbf{v}|f_\ell,f_h]+J_{\ell\ell}[\mathbf{v}|f_\ell,f_\ell]. \tag{1}$$

Multiplying by v^2 and integrating over velocity one gets

$$\partial_t\langle v_h^2\rangle=-(\zeta_{hh}+\zeta_{h\ell})\langle v_h^2\rangle,\quad \partial_t\langle v_\ell^2\rangle=-(\zeta_{\ell\ell}+\zeta_{\ell h})\langle v_\ell^2\rangle,\quad \zeta_{ij}=\frac{\int d\mathbf{v}\,v^2 J_{ij}[\mathbf{v}|f_i,f_j]}{\int d\mathbf{v}\,v^2 f_i(\mathbf{v})}. \tag{2}$$

The diagonal terms ζ_{hh} and $\zeta_{\ell\ell}$ are positive definite and represent the inelastic cooling rates of species h and ℓ, respectively. On the other hand, the cross term $\zeta_{h\ell}$ represents the "thermalization" rate of species h due to collisions with particles of species ℓ and can be either positive or negative, depending on the state of the mixture. Analogously, the term $\zeta_{\ell h}$ represents the thermalization rate of species ℓ due to collisions with particles of species h.

The four rates ζ_{ij} are complicated nonlinear functionals of the unknown distribution functions f_h and f_ℓ. However, reasonably good estimates can be obtained by using the Maxwellian approximation

$$f_i(\mathbf{v}) = n_i\left(m_i/2\pi T_i\right)^{3/2}\exp\left(-m_i v^2/2T_i\right), \quad T_i = m_i\langle v_i^2\rangle/3. \tag{3}$$

This approximation yields $\zeta_{ij} = \omega\xi_{ij}$, where $\omega = 4\pi n\chi_{h\ell}\sigma_{h\ell}^2\langle v_\ell\rangle(1+\alpha_{h\ell})m_\ell/3(m_h+m_\ell)$ is an effective collision frequency and the dimensionless cooling and thermalization rates are [1]

$$\xi_{hh}(\phi) = x_h\sqrt{\phi}\beta_h, \quad \xi_{h\ell}(\phi) = x_\ell\frac{\sqrt{1+\phi}}{1+\phi_0}\left(1-\mu+\phi_0-\frac{\mu}{\phi}\right), \tag{4}$$

$$\xi_{\ell\ell}(\phi) = x_\ell\beta_\ell, \quad \xi_{\ell h}(\phi) = x_h\frac{\sqrt{1+\phi}}{1+\phi_0}\left(1+\frac{\phi_0-\phi}{\mu}+\phi\right)(1-\mu). \tag{5}$$

Here, $\phi \equiv \langle v_h^2\rangle/\langle v_\ell^2\rangle$ is the mean square velocity ratio, $\mu \equiv m_\ell/(m_h+m_\ell)$ is the mass ratio, $\phi_0 \equiv (1-\alpha_{h\ell})/(1+\alpha_{h\ell})$ is a measure of the cross coefficient of restitution, and the coefficients

$$\beta_h \equiv \frac{1+\phi_0}{4\sqrt{2}}\frac{1-\alpha_{hh}^2}{\mu}\frac{\chi_{hh}}{\chi_{h\ell}}\left(\frac{\sigma_h}{\sigma_{h\ell}}\right)^2, \quad \beta_\ell \equiv \frac{1+\phi_0}{4\sqrt{2}}\frac{1-\alpha_{\ell\ell}^2}{\mu}\frac{\chi_{\ell\ell}}{\chi_{h\ell}}\left(\frac{\sigma_\ell}{\sigma_{h\ell}}\right)^2 \tag{6}$$

essentially measure the self coefficients of restitution and are inversely proportional to μ. The evolution equations (2) yield $\omega^{-1}\partial_t\phi = (\xi_{\ell\ell}+\xi_{\ell h}-\xi_{hh}-\xi_{h\ell})\phi$, so that the homogeneous cooling state of the mixture is characterized by the stationarity condition

$$\xi_{hh}(\phi)+\xi_{h\ell}(\phi)-\xi_{\ell\ell}(\phi)-\xi_{\ell h}(\phi) = 0. \tag{7}$$

This equation can be transformed into a tenth-degree algebraic equation for ϕ. Its physical solution determines ϕ as a function of the five control parameters of the problem, namely $\{x_h, \alpha_{hh}, \alpha_{\ell\ell}, \alpha_{h\ell}, m_\ell/m_h\}$ or, equivalently, $\{x_h, \beta_h, \beta_\ell, \phi_0, \mu\}$.

Equipartition of energy implies that $\phi = \mu/(1-\mu)$. Of course, this happens when all the collisions are elastic (i.e., $\beta_h = \beta_\ell = \phi_0 = 0$). When collisions are inelastic, the most common situation is $\phi > \mu/(1-\mu)$, i.e., the heavy particles have a larger (partial) temperature than the light particles, $T_h > T_\ell$ [3, 4]. However, this is not always necessarily the case. To clarify this point, it is convenient to define the parameters $K_i \equiv \mu\beta_i(1+\phi_0)/\phi_0$, $i = h, \ell$, which essentially measure the degree of inelasticity of the i-i collisions, relative to the inelasticity of the h-ℓ collisions. A careful analysis of Eq. (7) shows that, for a given value $\mu < \frac{1}{2}$, one has $T_h > T_\ell$ when one of the following three alternative cases occurs: (i) $K_h > K_h^*(\mu)$, $K_\ell > K_\ell^*(\mu)$, and $x_h < x_h^*(K_h, K_\ell, \mu)$; (ii) $K_h < K_h^*(\mu)$, $K_\ell < K_\ell^*(\mu)$, and $x_h > x_h^*(K_h, K_\ell, \mu)$; (iii) $K_h < K_h^*(\mu)$ and $K_\ell > K_\ell^*(\mu)$. Here,

$$K_h^*(\mu) \equiv \frac{1-\mu}{\sqrt{\mu}}, \quad K_\ell^*(\mu) \equiv \frac{\mu}{\sqrt{1-\mu}}, \quad x_h^*(K_h, K_\ell, \mu) \equiv \frac{K_\ell\sqrt{1-\mu}-\mu}{K_h\sqrt{\mu}+K_\ell\sqrt{1-\mu}-1}. \tag{8}$$

The left panel of Fig. 1 sketches the regions in the K_h–K_ℓ plane where $T_h > T_\ell$ or $T_h < T_\ell$ for a given mass ratio. The right panel shows the temperature ratio versus concentration for a few representative cases.

THE DISPARATE-MASS LIMIT. PHASE DIAGRAMS

We are now interested in the limit $m_\ell/m_h \to 0 \Rightarrow \mu \to 0$. The question we want to address is, how does the solution ϕ of Eq. (7) behave in the limit $\mu \to 0$? There are two distinct possibilities: either $\lim_{\mu\to 0}\phi = 0$ or $\lim_{\mu\to 0}\phi \neq 0$. Thus, ϕ can be viewed as an "order" parameter. The case $\lim_{\mu\to 0}\phi = 0$ means that the heavy particles move much more slowly than the light particles; we will say that the mixture is then in a *normal* state. The alternative case $\lim_{\mu\to 0}\phi \neq 0$ implies that the speed of the heavy particles is typically comparable (or even larger) than that of the light species; we will call *ordered* to such a state.

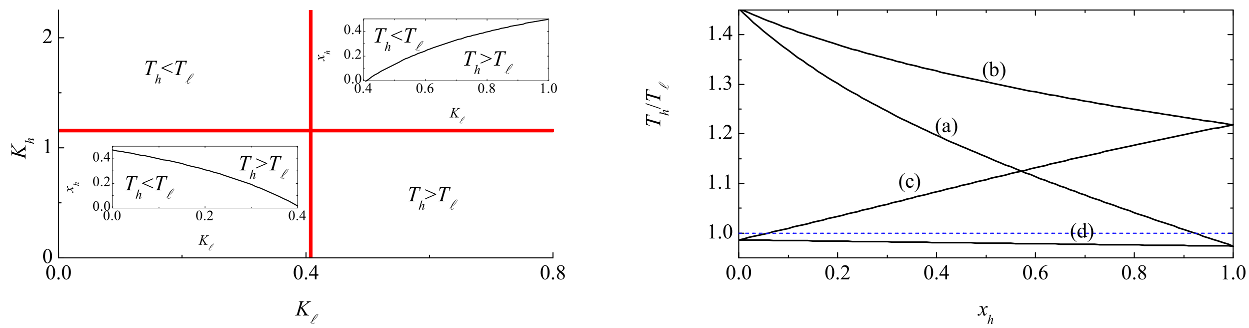

FIGURE 1. Left panel: Plane K_h vs K_ℓ split into four quadrants by the lines $K_h = K_h^*(\mu)$ and $K_\ell = K_\ell^*(\mu)$ for $\mu = \frac{1}{3}$; in the top-right quadrant, $T_h > T_\ell$ if $x_h < x_h^*(K_h, K_\ell, \mu)$, as represented by the inset, where x_h^* is plotted as a function of K_ℓ for $\mu = \frac{1}{3}$ and $K_h = 2$; in the bottom-right quadrant, $T_h > T_\ell$ for any concentration; in the bottom-left quadrant, $T_h > T_\ell$ if $x_h > x_h^*(K_h, K_\ell, \mu)$, as represented by the inset, where x_h^* is plotted as a function of K_ℓ for $\mu = \frac{1}{3}$ and $K_h = 0.5$; finally, in the top-left quadrant, $T_h < T_\ell$ for any concentration. Right panel: Plot of the temperature ratio T_h/T_ℓ versus x_h for $\alpha_{h\ell} = 0.8$, $m_h = 2m_\ell$, $\sigma_h = \sigma_\ell$, and (a) $(\alpha_{hh}, \alpha_{\ell\ell}) = (0.6, 0.6)$, (b) $(\alpha_{hh}, \alpha_{\ell\ell}) = (0.9, 0.6)$, (c) $(\alpha_{hh}, \alpha_{\ell\ell}) = (0.9, 0.9)$, and (d) $(\alpha_{hh}, \alpha_{\ell\ell}) = (0.6, 0.9)$.

In order to make a finer classification, let us introduce a "critical exponent" η such that $\phi \sim \mu^\eta$ as $\mu \to 0$, so $\eta > 0$ corresponds to normal states and $\eta \leq 0$ correspond to ordered states. If $\eta = 1$, the equipartition of energy is weakly broken in the sense that $T_h/T_\ell =$ finite, as in the elastic case. This corresponds to a subclass of the normal state that we will call *partitioned*. An example of this situation occurs when the self-collisions are elastic and the cross collisions are quasi-elastic, with $1 - \alpha_{h\ell} \sim \mu$. On the other hand, if $\eta > 1$, then ϕ decays even more rapidly than in the elastic limit. Again, this is a normal state ($\phi \to 0$) but is not partitioned ($T_h/T_\ell \to 0$) since all the energy is carried by the light particles. This case will be referred to as a *mono-energetic* ℓ state, an example of which taking place when only the *h*-*h* collisions are inelastic. Still in the normal class, the case $0 < \eta < 1$ represents a breakdown of equipartition intermediate between that of the partitioned state and that of the ordered state: while the heavy particles move much more slowly than the light ones, they carry all the energy of the mixture. We will refer to this situation as a *mono-energetic h* state. A simple example corresponds to mixtures of disparate sizes when only the ℓ-ℓ collisions are inelastic. Thus, three subclasses (mono-energetic ℓ, partitioned, and mono-energetic h) can be distinguished within the class of normal states. As for ordered states, the typical case corresponds to $\eta = 0$, i.e., $\phi =$ finite in the disparate-mass limit. This happens, for instance, when only the cross collisions are inelastic, regardless of the values of the mole fractions. In particular, the state is ordered both in the "Brownian" limit $x_h \to 0$ and in the "Lorentz" limit $x_\ell \to 0$. In the former case, $\phi \neq 0$ implies that the *dynamics* is not Brownian since the solute Brownian particles move with a speed comparable to that of the fluid particles. In the latter case, the system does not behave as a Lorentz gas since the heavy "scatterers" are never at rest. An even stronger breakdown of energy equipartition occurs if $\eta < 0$, implying not only $\phi \neq 0$ (ordered state), but $\phi \to \infty$. We will say that in this case the system reaches a *super-ordered* state. A necessary condition for the existence of this situation is the Brownian limit $x_h \to 0$. A super-ordered state appears, for instance, if only the ℓ-ℓ collisions are inelastic and $x_h \sim \mu$, so that the total masses of both species are comparable.

TABLE 1. Possible classes of states in a binary granular mixture in the disparate-mass limit $m_\ell/m_h \approx \mu \to 0$. The mean square ratio $\phi = \langle v_h^2\rangle/\langle v_\ell^2\rangle$ scales as $\phi \sim \mu^\eta$.

Class	**Subclass**	η	$\langle v_h^2\rangle/\langle v_\ell^2\rangle$	T_h/T_ℓ	**Example**
Normal	Mono-energetic ℓ	$\eta > 1$	0	0	$\alpha_{hh} < 1$
	Partitioned	$\eta = 1$	0	finite	$1 - \alpha_{h\ell} \sim m_\ell/m_h$
	Mono-energetic h	$0 < \eta < 1$	0	∞	$\alpha_{\ell\ell} < 1, \sigma_i \sim m_i^{1/3}$
Ordered	Ordered	$\eta = 0$	finite	∞	$\alpha_{h\ell} < 1$
	Super-ordered	$\eta < 0$	∞	∞	$\alpha_{\ell\ell} < 1, x_h \sim m_\ell/m_h$

Table 1 describes the five possible subclasses (or phases) and the corresponding representative examples. In general, depending on the regime of values for the control parameters α_{ij} and x_h, the asymptotic homogeneous cooling state of a disparate-mass binary mixture belongs in one of those subclasses. In order to investigate the regions in the parameter

space corresponding to each phase, let us assume scaling laws of the form

$$\beta_h \sim \mu^{-1+a_h}, \quad \beta_\ell \sim \mu^{-1+a_\ell}, \quad \phi_0 \sim \mu^b, \quad x_h \sim \mu^{c_h}, \tag{9}$$

where the exponents a_h, a_ℓ, b, and c_h are non-negative. The exponents a_h and b measure, respectively, the inelasticity of the h-h and h-ℓ collisions (i.e., $1-\alpha_{hh} \sim \mu^{a_h}$, $1-\alpha_{h\ell} \sim \mu^b$), while a_ℓ measures the inelasticity of the ℓ-ℓ collisions, as well as the possible dependence of the size ratio on the mass ratio: $(1-\alpha_{\ell\ell})(\sigma_\ell/\sigma_h)^2 \sim \mu^{a_\ell}$. If any of the three types of collision is elastic, the corresponding exponent takes an infinite value. Conversely, if the collision is inelastic then the exponent vanishes. A finite value of a_h or b implies that the associated collision is quasi-elastic. In the case $a_\ell =$ finite, either the ℓ-ℓ collisions are quasi-elastic or the size ratio scales with the mass ratio, or both. For finite concentrations of the h species one has $c_h = 0$, while $c_h > 0$ in the Brownian limit ($x_h \to 0$). From Eqs. (4) and (5), the scaling behaviors of ξ_{ij} become

$$\xi_{hh} \sim \mu^{-1+a_h+c_h+\eta/2}, \quad \xi_{h\ell} \sim \max\{\mu^0, \mu^{\eta/2}\}\max\{\mu^0, |\mu^{1-\eta}|\}, \tag{10}$$

$$\xi_{\ell\ell} \sim \mu^{-1+a_\ell}, \quad \xi_{\ell h} \sim \mu^{c_h}\max\{\mu^0, \mu^{\eta/2}\}\max\{\mu^0, \mu^{-1+b}, |\mu^{-1+\eta}|\}. \tag{11}$$

In the above equations, $|\cdots|$ indicates that the corresponding term has a negative prefactor.

Let us first assume that the mole fraction x_h is kept finite in the limit $\mu \to 0$, so that $c_h = 0$. In that case, condition (7) is fulfilled in the limit $\mu \to 0$ when any of the five conditions described in Table 2 is satisfied. The associated constraints and expressions for the exponent η are also included in Table 2. It is easy to check that in region B one always has $\eta > 1$, while in region D one always has $0 < \eta < 1$. In region E, $\eta = 1$ if $b \geq 1$, while $0 < \eta < 1$ if $0 < b < 1$. As for region A, $0 < \eta < 1$ if $a_\ell < a_h + \frac{1}{2}$, while $\eta > 1$ otherwise, this latter case being only possible when $b \geq \frac{1}{2}$. Likewise, in region C one has $0 < \eta < 1$ if $a_h > b - \frac{1}{2}$ and $\eta > 1$ otherwise, this latter case being only possible when $b \geq \frac{1}{2}$. The phase diagrams are shown in Fig. 2.

TABLE 2. Possible cases in the disparate-mass limit at finite concentration ($c_h = 0$).

Case	Conditions	Constraints	η
A	$\xi_{hh} \sim \xi_{\ell\ell} \gg \lvert\xi_{h\ell}\rvert, \lvert\xi_{\ell h}\rvert$	$2a_h < a_\ell < \min\{b, \frac{2}{3}(a_h+1)\}$	$2(a_\ell - a_h)$
B	$\xi_{hh} \sim \lvert\xi_{h\ell}\rvert \gg \xi_{\ell\ell}, \lvert\xi_{\ell h}\rvert$	$a_\ell > \frac{2}{3}(a_h+1)$, $a_h < \min\{\frac{1}{2}, \frac{3}{2}b - 1\}$	$\frac{2}{3}(2-a_h)$
C	$\xi_{hh} \sim \lvert\xi_{\ell h}\rvert \gg \xi_{\ell\ell}, \lvert\xi_{h\ell}\rvert$	$b < 1$, $a_\ell > b$, $\frac{3}{2}b - 1 < a_h < \frac{1}{2}b$	$2(b - a_h)$
D	$\xi_{\ell\ell} \sim \lvert\xi_{\ell h}\rvert \gg \xi_{hh}, \lvert\xi_{h\ell}\rvert$	$a_\ell < \min\{2a_h, b, 1\}$	a_ℓ
E	$\lvert\xi_{\ell h}\rvert \gg \xi_{\ell\ell}, \xi_{hh}, \lvert\xi_{h\ell}\rvert$	$a_h > \frac{1}{2}\min\{b,1\}$, $a_\ell > \min\{b,1\}$	$\min\{b,1\}$

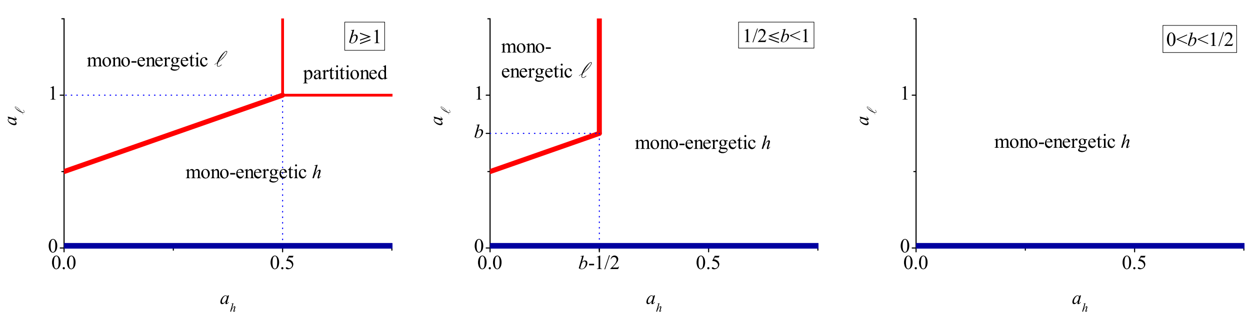

FIGURE 2. Regions in the plane a_ℓ vs a_h corresponding to each phase. The thick lines separating the phases mono-energetic ℓ and mono-energetic h correspond to partitioned states. The thick line at $a_\ell = 0$ represents ordered states. The whole plane is occupied by ordered states if $b = 0$ (not shown).

Now we assume that the concentration of h particles behaves as $x_h \sim \mu^{c_h}$ in the limit $\mu \to 0$. The total mass of the h species is larger than that of the ℓ species (i.e., $x_h m_h \gg x_\ell m_\ell$) if $c_h < 1$, while the opposite happens if $c_h > 1$. This limit adds a new parameter $c_h > 0$ to the parameter space, so that the phase diagrams in Fig. 2 are modified. Let us restrict ourselves to $b = 0$, i.e., the cross collisions are kept inelastic in the limit $\mu \to 0$. As indicated in the caption of Fig. 2, if $b = 0$ the state of the mixture is always ordered ($\eta = 0$) in the case of finite concentrations ($c_h = 0$). However, this uniform situation changes dramatically if $c_h > 0$. With $b = 0$, it is easy to realize from Eqs. (10) and (11) that

$|\xi_{\ell h}|$ is never negligible versus ξ_{hh}. That means that the value of the parameter a_h (i.e., the degree of inelasticity of the h-h collisions) will not play any role, so the parameter space reduces to a_ℓ and c_h. The possible cases are described in Table 3. Cases G_1 and G_2 correspond to $\max\{\mu^0,\mu^{\eta/2}\}=\mu^{\eta/2}$ and $\max\{\mu^0,\mu^{\eta/2}\}=\mu^0$, respectively. While G_1 defines a whole region in the parameter space with a well-defined (negative) expression for the exponent η, G_2 defines a line along which the value of η is not unique. A similar situation occurs with case H. The phase diagram is shown in Fig. 3. We observe that η changes discontinuously from $\eta=0$ to $\eta=1$ when going from the ordered region to the partitioned region through line H. Even more discontinuous is the transition from the super-ordered region ($\eta<0$) to the partitioned region ($\eta=1$) through line G_2. These are *critical* lines along which the value of η is not uniquely determined by the values of a_ℓ and c_h. Let us analyze these critical lines with some detail.

TABLE 3. Possible cases in the disparate-mass limit with $x_h\to 0$ and $\alpha_{h\ell}<1$ ($b=0$).

Case	Conditions	Constraints	η
F	$\vert\xi_{h\ell}\vert\gg\xi_{\ell\ell},\vert\xi_{\ell h}\vert$	$a_\ell>1$, $c_h>1$	1
G_1	$\vert\xi_{h\ell}\vert\sim\xi_{\ell\ell}\gg\vert\xi_{\ell h}\vert$	$\frac{1}{2}(3-c_h)<a_\ell<1$, $c_h>1$	$2(a_\ell-1)$
G_2	$\vert\xi_{h\ell}\vert\sim\xi_{\ell\ell}\gg\vert\xi_{\ell h}\vert$	$a_\ell=1$, $c_h>1$	$0\le\eta\le 1$
H	$\vert\xi_{h\ell}\vert\sim\vert\xi_{\ell h}\vert\gg\xi_{\ell\ell}$	$c_h=1$, $a_\ell>1$	$0\le\eta\le 1$
I	$\xi_{\ell\ell}\sim\vert\xi_{\ell h}\vert\gg\vert\xi_{h\ell}\vert$	$a_\ell<\min\{c_h,\frac{1}{2}(3-c_h)\}$	$\frac{2}{3}(a_\ell-c_h)$
J	$\vert\xi_{\ell h}\vert\gg\xi_{\ell\ell},\vert\xi_{h\ell}\vert$	$a_\ell>c_h$, $c_h<1$	0

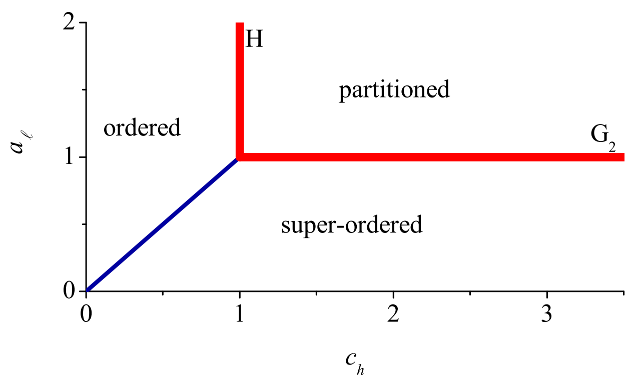

FIGURE 3. Regions in the plane a_ℓ vs c_h corresponding to the phases present in the limit of vanishing concentration of heavy particles when $1-\alpha_{h\ell}$ is kept finite ($b=0$). The thick line separating the regions super-ordered and partitioned (line G_2) and the thick line separating the regions ordered and partitioned (line H) are critical lines on which the state can be partitioned, mono-energetic h, or ordered, depending on the value of the parameters β_ℓ (line G_2) or γ (line H).

Along the critical line G_2 the leading terms are $\xi_{h\ell}$ and $\xi_{\ell\ell}$, so that condition (7) becomes

$$\sqrt{1+\phi}\left[1-\frac{\mu}{\phi(1+\phi_0)}\right]=\beta_\ell. \tag{12}$$

The solution to this equation in the limit $\mu\to 0$ is

$$\phi=\begin{cases}\frac{1}{1-\beta_\ell}\frac{\mu}{1+\phi_0} & \text{if } \beta_\ell<1,\\ \sqrt{\frac{2\mu}{1+\phi_0}} & \text{if } \beta_\ell=1,\\ \beta_\ell^2-1 & \text{if } \beta_\ell>1.\end{cases} \tag{13}$$

Therefore, β_ℓ plays the role of a *control* parameter and $\beta_\ell=1$ is its critical value. If $\beta_\ell<1$ the state is partitioned, if $\beta_\ell=1$ it is mono-energetic h, and if $\beta_\ell>1$ it is ordered. This situation has been analyzed in detail in Ref. [5]. Since $c_h>1$, the total mass of the h species is negligible versus that of the ℓ species. Moreover, the ℓ particles are not affected by the presence of the h particles.

Along the critical line H, $x_h\sim\mu$ (i.e., $c_h=1$), what means that both species have comparable total masses. In addition, the ℓ-ℓ collisions are not too inelastic (or the size ratio σ_ℓ/σ_h is sufficiently small), in the sense that

$(1-\alpha_{\ell\ell}^2)(\sigma_\ell/\sigma_h)^2 \sim \mu^{a_\ell}$ with $a_\ell > 1$. In that situation, the leading terms are $\xi_{h\ell}$ and $\xi_{\ell h}$. Therefore, condition (7) yields

$$1-\frac{\mu}{\phi(1+\phi_0)}=\frac{x_h}{\mu}\frac{\phi_0-\phi}{1+\phi_0}, \tag{14}$$

whose solution is

$$\phi=\begin{cases}\frac{1}{1-\gamma}\frac{\mu}{1+\phi_0} & \text{if } \gamma<1,\\ \sqrt{\frac{\phi_0\mu}{1+\phi_0}} & \text{if } \gamma=1,\\ \frac{\phi_0}{\gamma}(\gamma-1) & \text{if } \gamma>1.\end{cases} \tag{15}$$

Here, $\gamma\equiv(x_h/\mu)\phi_0/(1+\phi_0)=(x_h/\mu)(1-\alpha_{h\ell})/2$, which represents the control parameter in this case.

CONCLUSIONS

In this paper we have shown that in the homogeneous cooling state of a granular binary mixture the mean square velocity ratio $\phi=\langle v_h^2\rangle/\langle v_\ell^2\rangle$ and the temperature ratio T_h/T_ℓ can take widely different values depending on the parameters of the system (coefficients of restitution, concentrations, and mass and size ratios). Typically, $T_h > T_\ell$ if (i) the self-collisions (h-h and ℓ-ℓ) are sufficiently more inelastic than the cross collisions (h-ℓ) and the concentration of the h particles is sufficiently low; (ii) the h-h and ℓ-ℓ collisions are sufficiently less inelastic than the h-ℓ collisions and x_h is large enough; (iii) the h-h collisions are not too inelastic but the ℓ-ℓ collisions are sufficiently inelastic.

In the disparate-mass limit $\mu\to 0$, the breakdown of equipartition can become extreme with divergent, finite, or vanishing limits of the mean square velocity and/or temperature ratios, depending on the scaling behavior of the parameters. This gives rise to the five classes of states (or phases) described in Table 1, ranging from the mono-energetic ℓ state ($T_h/T_\ell\to 0$) to the super-ordered state ($\phi\to\infty$). The associated phase diagrams are presented in Figs. 2 and 3. If the cross collisions are inelastic ($\alpha_{h\ell}<1$), the state is always ordered ($\phi=$ finite). As a consequence, in this case there is neither Brownian dynamics (when $x_h\to 0$) nor Lorentz dynamics (when $x_\ell\to 0$). A partitioned state ($T_h/T_\ell=$ finite) is only possible if the three types of collisions are sufficiently quasi-elastic, while a super-ordered state ($\phi\to\infty$) is only possible in the Brownian limit ($x_h\to 0$). In that limit, there is no mono-energetic ℓ state and there exist critical lines in the phase diagram where the state can be partitioned, ordered, or mono-energetic h.

The analysis performed in this work has been restricted to the homogeneous cooling state, but it can be easily extended to mixtures heated with a white-noise forcing [2]. In that case, condition (7) is replaced by $\phi\,[\xi_{hh}(\phi)+\xi_{h\ell}(\phi)]-\xi_{\ell\ell}(\phi)-\xi_{\ell h}(\phi)=0$. As a consequence, the phase monoenergetic ℓ is suppressed.

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